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GROUPS WITH MAXIMAL SUBGROUPS OF SYLOW SUBGROUPS NORMAL

BY
GARY L. WALLS

ABSTRACT

This paper characterizes those finite groups with the property that maximal subgroups of Sylow subgroups are normal. They are all certain extensions of nilpotent groups by cyclic groups.

In his paper [2] S. Srinivasan studied groups in which maximal subgroups of Sylow subgroups are normal subgroups. We call such groups MNP-groups and are able to obtain a classification of such groups.

Srinivasan [2] showed that MNP-groups were supersolvable. As all nilpotent groups are MNP-groups we are studying a class of groups strictly between the nilpotent and the supersolvable groups.

The following facts follow immediately from the definition.

LEMMA 1. *A Hall subgroup of an* MNP-*group must be an* MNP-*group.*

2. *The direct product of* MNP-*groups of relatively prime orders are* MNP-*groups.* [Note: $S_3 \times Z_2$ shows the necessity of the condition on the orders.]

3. *A Sylow p-subgroup of an* MNP-*group must be either cyclic or normal.*

The last part follows from the fact that a group with a unique maximal subgroup must be cyclic. Now if we let H be the product of all the normal Sylow subgroups of G, we can write $G = HK$ where K is the complement of H in G guaranteed by the Schur–Zassenhaus theorem.

In this factorization of G we see that H is nilpotent and K is a group all of whose Sylow subgroups are cyclic. The problem has now become to (1) determine the possible structure of K, and then (2) determine the action of K on H. We recall (cf. Scott [1], theorem 12.6.17) that a non-cyclic group K with all Sylow subgroups cyclic has the following presentation:

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$$\langle a, b \mid a^m = 1, b^n = 1, b^{-1}ab = a^r \rangle$$

$$\text{where } (m, n) = 1 = (m, r-1), \quad r^n \equiv 1 \pmod{m}.$$

THEOREM 4. *Let K be a group all of whose Sylow subgroups are cyclic. Then K is an* MNP-*group if and only if* $K = \langle a, b \rangle$, $|a| = m$, $|b| = n = p_1^{r_1} \cdots p_s^{r_s}$, $(m, n) = 1$, $b = b_1 \cdots b_s$ *with* $|b_i| = p_i^{r_i}$ *and* b_i *acts as an automorphism of order* $\leqq p_i$ *on* $\langle a \rangle$.

PROOF. ($\Rightarrow$) K has the above form, as $\langle b_i \rangle$ is a Sylow p_i-subgroup of G, $\langle b_i^{p_i} \rangle \lhd K$. Hence, $[b_i^{p_i}, \langle a \rangle] = 1$.

($\Leftarrow$) This direction follows from the fact that any Sylow subgroup of G is conjugate to a subgroup of $\langle a \rangle$ or $\langle b \rangle$.

We note the following general fact.

LEMMA 5. *Suppose* $G = P\langle x \rangle$, P *a normal* p-*subgroup of* G *and* x *is a* q-*element. Then, if* G *is an* MNP-*group, the automorphism that* x *induces on* $P/\Phi(P)$ *is potentiation (raising to powers).*

PROOF. Now all the maximal subgroups of $P/\Phi(P)$ must be normalized by x. It follows that all the subgroups of $P/\Phi(P)$ in $[P/\Phi(P)]\langle x \rangle$ (the semi-direct product) are normal. Let $P/\Phi(P) = \langle a_1 \rangle \times \cdots \times \langle a_k \rangle$. Now $\langle a_1 \rangle$, $\langle a_2 \rangle$, $\langle a_1 a_2 \rangle \lhd P/\Phi(P)$, so $x^{-1}a_1x = a_1^l$, $x^{-1}a_2x = a_2^m$ and $x^{-1}(a_1a_2)x = (a_1a_2)^n = a_1^n a_2^n = a_1^l a_2^m$. If follows that

$$a_1^{n-l} = a_2^{m-n} \in \langle a_1 \rangle \cap \langle a_2 \rangle = 1.$$

Thus, $n \equiv l \pmod{p}$ and $m \equiv n \pmod{p}$. Hence, $l \equiv m \pmod{p}$. It follows that for all $a \in P/\Phi(P)$, $x^{-1}ax = a^l$.

The above lemma leads to the following characterization:

THEOREM 6. *Let G be a finite group. Then, G is an* MNP-*group if and only if* $G = H\langle x \rangle$ *where* (i) H *is a normal, nilpotent Hall subgroup of* G, (ii) *the generators of Sylow subgroups of* $\langle x \rangle$ *induce automorphisms of order dividing a prime on* H *which act as potentiation on* $H/\Phi(H)$.

PROOF. ($\Leftarrow$) This direction is clear.

($\Leftarrow$) By mathematical induction and Theorem 4 we may assume that $G = P\langle x, y \rangle$ with $P \lhd G$, a p-group, $\Phi(P) = 1$, $|x| = q^n$, $|y| = s$, $y^{-1}xy = x^r$, $r^s \equiv 1 \pmod{q^n}$, $(q, r-1) = (q, s) = 1$, $p \nmid |\langle x, y \rangle|$.

Now from Lemma 5, x, y act as potentiation on P. Thus, for $z \in P$, $y^{-1}zy = z^{l_y}$ and $x^{-1}zx = z^{l_x}$.

If $l_x = 1$, then $G = (P \times \langle x \rangle)\langle y \rangle$ has the desired form by the Chinese Remainder theorem.

If $l_x > 1$, then since $y^{-1}xy = x^r$, it follows from their actions on z, that $z^{l_x^r} = z^{l_x}$. Hence, $p \mid l_x^r - l_x$. As $(p, l_x) = 1$, $l_x^{r-1} \equiv 1 \pmod{p}$. However, as $[x^q, P] = 1$, $l_x^q \equiv 1 \pmod{p}$. Thus, $q \mid r - 1$ contrary to the assumption that $(q, r-1) = 1$. This really means that $\langle x, y \rangle$ was actually cyclic and so G must have the desired form.

References

1. W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

2. S. Srinivasan, *Two sufficient conditions for supersolvability of finite groups*, Isr. J. Math. **35** (1980), 210–214.

Department of Mathematics
The University of Southern Mississippi
Hattiesburg, MS 39401 USA